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1987 J. Phys. A: Math. Gen. 20 L29

(http://iopscience.iop.org/0305-4470/20/1/007)

J. Phys. A: Math. Gen. 20 (1987) L29-L33. Printed in the UK

LETTER TO THE EDITOR

How long are the arms in DLA?

Harry Kesten
Department of Mathematics, Cornell University, Ithaca, NY 14853, USA

Received 20 October 1986

Abstract. We show that the maximal length of the arms of DLA can grow at most at a rate of $n^{2/3}$ in dimension 2 and at a rate of $n^{2/d}$ in dimension $d > 2$. Here n denotes the volume or mass of the aggregate.

Much attention has been paid in the last few years to the growth model known as diffusion-limited aggregation (DLA) (see for instance the many articles in Stanley and Ostrowsky (1986) (in particular Witten (1986) and Meakin (1986))) and the many talks at the 1986 *Statistical Physics Meeting* in Boston. We discuss here only the most classical of the DLA models, the Witten-Sander model on the hypercubic lattice z^d (Witten and Sander 1981). We denote the aggregate of n particles by A_n. A_1 consists only of the origin $\mathbf{0}$ in z^d. In the Witten-Sander model A_{n+1} is formed from A_n by releasing a particle 'at infinity' and letting it perform a (nearest-neighbour) symmetric random walk on z^d until it reaches a boundary site of A_n. (A boundary site of A_n is a vertex of z^d adjacent to A_n, but not in A_n.) If y is the first boundary site visited by the particle, then $A_{n+1} = A_n \cup \{y\}$, i.e. A_{n+1} is formed by adding y to A_n. We shall give a more formal description in remark 3, but first we state our result.

Theorem. Let

$$r_n := \max\{|x|: x \in A_n\}$$

denote the 'radius' of A_n. There exists a constant C_d, depending only on the dimension d, such that

$$\frac{1}{n^{2/3}} r_n \leq C_2 \qquad \text{eventually with probability 1 in dimension 2} \tag{1}$$

and

$$\frac{1}{n^{2/d}} r_n \leq C_d \qquad \text{eventually with probability 1 in dimension } d > 2. \tag{2}$$

Remark 1. Family (1986) has shown that if A_n looks like a cross with needlelike arms along the coordinate axes in $d = 2$ then the length of the arms of A_n should grow at the rate $n^{2/3}$. Turkevich (1986) assumed that A_n was diamondlike, which resulted in a growth rate of $n^{3/5}$ for r_n in dimension two. Here we make *no a priori assumptions on the shape of* A_n. It will be seen later that our estimates are quite crude, so that it seems quite possible to us that the actual growth rate of r_n is smaller than $n^{2/3}$ for $d = 2$.

Remark 2. Various people (cf Witten (1986, pp 65, 66), Meakin (1986, p 120) and the references cited there) have argued that, for large d, r_n should grow at most like $n^{1/(d-1)}$, and perhaps for $d=2, 3, 4$ like $n^{(d+1)/(d^2+1)}$. Clearly (2) is a much worse estimate.

Remark 3. The description of the formation of A_{n+1} from A_n was somewhat informal, in that we cannot release a particle at infinity. Moreover, for $d \geq 3$ such a particle will never hit the boundary of A_n. To circumvent these difficulties we take limits of hitting probabilities. Let $S_0, S_1, \ldots$ be the successive positions of a particle performing a nearest-neighbour symmetric random walk on z^d (starting at S_0). Let B be a collection of lattice sites and let $T = T(B)$ be the *hitting time* of B, i.e.

$$T(B) = \min\{n \geq 0 \colon S_n \in B\}$$

($T(B) = \infty$ if S_n never is in B). The *hitting position* of B is therefore $S_{T(B)}$. The *hitting distribution* is given by

$$H_B(x, y) := \Pr\{S_{T(B)} = y \mid S_0 = x\} \qquad y \in B.$$

It is known (Spitzer 1976, theorem 14.1) that for $d = 2$ and any finite B

$$\mu_B(y) := \lim_{|x| \to \infty} H_B(x, y) \qquad \text{exists} \qquad y \in B \tag{3}$$

$$\sum_{y \in B} \mu_B(y) = 1. \tag{4}$$

Thus μ_B defines an honest probability distribution. For $d = 2$ we apply this with $B = \partial A_n :=$ boundary of A_n. We form A_{n+1} by adding to A_n a site y_{n+1}, where y_{n+1} is chosen according to the distribution $\mu_{\partial A_n}$.

For $d \geq 3$ the limit of $H_B(x, y)$ for $|x| \to \infty$ is identically zero (Spitzer 1976, prop. 25.3). One obtains a non-trivial limit distribution for y by conditioning on $\{T_B < \infty\}$, the event that B is visited at some time. In fact, for $d \geq 3$ the limit

$$\begin{aligned} \mu_B(y) &:= \lim_{|x| \to \infty} H_B(x, y) \left(\sum_{Z \in B} H(x, z) \right)^{-1} \\ &= \lim_{|x| \to \infty} \Pr\{S_{T(B)} = y \mid S_0 = x, T(B) < \infty\} \end{aligned} \tag{5}$$

exists and satisfies (4) (cf Spitzer (1976, prop. 26.2) when $d = 3$; the same proof works for $d > 3$). Again we choose the point y_{n+1}, to be added to A_n, according to the distribution $\mu_{\partial A_n}$.

In order to prove this, first one derives a uniform upper bound for $\mu_{\partial A_n}(y)$ in terms of r_n or n, and then one shows how this limits the growth rate as described in (1) and (2). We describe some more details in the following steps. Steps (i) and (ii) consider $d = 2$ only, while step (iii) discusses $d \geq 3$.

Step (i). Take $d = 2$ and let A_n, and hence ∂A_n be fixed, and also fix a point $y \in \partial A_n$. Then there exists a site z_0 in A_n which is adjacent to y and a path $z_0, z_1, \ldots, z_m$ of vertices of z^2 in A_n such that $|z_0 - z_m| \geq \frac{1}{2} r_n$ (z_{i+1} and z_i are adjacent on z^2, $0 \leq i < m$). This is so because A_n is connected and has radius r_n. Thus, there exists a path $z_0, z_1, \ldots, z_l$ in A_n from z_0 through $\mathbf{0}$ to some point z_l with $|z_l| = r_n$. Then $|z_0 - \mathbf{0}| \geq \frac{1}{2} r_n$ or $|z_0 - z_l| \geq \frac{1}{2} r_n$, so that we can take for $z_0, \ldots, z_m$ either the first part of $z_0, \ldots, z_l$ (connecting z_0 to $\mathbf{0}$) or the whole path $z_0, \ldots, z_l$. If the random walk particle starts outside A_n and hits ∂A_n first in y, then it actually hits y before it hits the path $\{z_0, z_1, \ldots, z_m\} \subset A_n$. Therefore, if we take $B = \{y, z_0, z_1, \ldots, z_m\}$, then

$$\mu_{\partial A_n}(y) \leq \mu_B(y). \tag{6}$$

Technically the hardest part of the proof is to show that, no matter what the path B with end-to-end distance $\geq \frac{1}{2}r_n$ is, $\mu_B(y)$ cannot be much larger than the hitting probability at $y = \mathbf{0}$, when B is a straight line segment along the negative x axis of length $\frac{1}{2}r_n$. Specifically, there exists a constant K^*, such that

$$\mu_B(y) \leq K^* \mu_C(\mathbf{0})$$

when

$$C = \{\mathbf{0}, (-1, 0), (-2, 0), \ldots, (-\tfrac{1}{2}r_n, 0)\}. \tag{7}$$

$\mu_C(\mathbf{0})$ can be estimated explicitly and (6) and (7) together result in

$$\mu_{\partial A_n}(y) \leq \frac{K}{\sqrt{r_n}} \tag{8}$$

for some fixed K. This estimate holds for all connected sets A_n with $\mathbf{0} \in A_n$ and radius r_n, and $y \in \partial A_n$.

We point out that the analogue of (7) for planar Brownian motion (even with $K^* = 1$) is a known inequality for harmonic measures. It is sometimes called the Beurling circular projection theorem (cf Ahlfors 1973, theorem 3.6). The rather technical proof of (7) which mimicks Beurling's proof will be given in a separate paper.

Step (ii). Equation (8) supplies us with a limit on the growth rate of r_n. In fact, we claim that with probability 1 there exists a random but finite k_0 such that

$$r(2^{k+1}) - r(l) \leq \frac{K 2^{k+4}}{\sqrt{r(l)}} + 2^{k/2} \qquad \text{for all } k \geq k_0, 2^k \leq l \leq 2^{k+1}. \tag{9}$$

Here we have written $r(n)$ instead of r_n for typographical reasons. Once one has (9) it is not too difficult to show that any sequence of positive r_n which is increasing and satisfies $r_{n+1} - r_n \leq 1$ as well as (9) must also satisfy (1) with $C_2 = 28(1 + 5K^{2/3})$.

Let $u_1, \ldots, u_s$ be a path without double points on z^2. We say that this path is *filled in order* if each u_i eventually belongs to A_n, and if $n_1 < n_2 < \ldots < n_s$, where n_i denotes the smallest n for which $u_i \in A_n$. To obtain (9) note that for any vertex $x \in A_n$ there exists a path $u_1 = \mathbf{0}$, $u_2, \ldots, u_s = x$, from $\mathbf{0}$ to x, in z^2 which is contained in A_n and filled in order (in particular $s \leq n$). The existence of such a path is easily established by induction on n. If we now take $n = 2^{k+1}$, $2^k \leq l < 2^{k+1}$, and $x \in A_{2^{k+1}}$ such that $|x| = r(2^{k+1})$, then the piece of the path leading to this x from its last crossing of the circle of radius r_l is a path without double points, u_t, $u_{t+1}, \ldots, u_s$, with the following properties:

$$r(l) < |u_t| \leq r(l) + 1 \tag{10}$$

$$r(l) < |u_i| \leq r(2^{k+1}) \qquad t \leq i \leq s \tag{11}$$

and

$$u_t, \ldots, u_s \text{ is filled in order during the time interval } [l+1, 2^{k+1}]. \tag{12}$$

If

$$r(2^{k+1}) - r(l) \geq \frac{K 2^{k+4}}{\sqrt{r(l)}} + 2^{k/2} + 1 \tag{13}$$

is to occur, then in addition we must have

$$s - t \geq \frac{K 2^{k+4}}{\sqrt{r(l)}} + 2^{k/2}. \tag{14}$$

Set

$$m=\frac{K2^{k+4}}{\sqrt{r(l)}}+2^{k/2}. \tag{15}$$

Then (13) can occur only if there exists a path $u_t, u_{t+1}, \ldots, u_{t+m}$ which satisfies (10)-(12) with s replaced by $t+m$. The number of paths of length m satisfying (10) and (11) is at most

$$4\pi^2(r_l+1)4^m. \tag{16}$$

Now fix a path $u_t, \ldots, u_{t+m}$ with the properties (10) and (11) for $k=t+m$. We shall estimate the probability of (12) for this path. Assume that at time $n \in [l+1, 2^{k+1}]$, A_{n-1} already contains $u_t, \ldots, u_\nu$, but not yet $u_{\nu+1}$. Define

$$I_n=\begin{cases}1 & \text{if } u_{\nu+1} \text{ is the vertex added to } A_{n-1} \text{ to form } A_n\\ 0 & \text{otherwise.}\end{cases}$$

I_n is the indicator function of a successful filling of a new site of the given path in order at time n. For (12) to occur it is necessary that

$$\sum_{l+1}^{2^{k+1}} I_n \geq m. \tag{17}$$

However, if we write $\Pr\{I_n=1 \mid \mathcal{F}_{n-1}\}$ for the conditional probability of $\{I_n=1\}$, given A_{n-1}, then (8) tells us that

$$\Pr\{I_n=1 \mid \mathcal{F}_{n-1}\} \leq \frac{K}{\sqrt{r(n-1)}} \leq \frac{K}{\sqrt{r(l)}} \qquad n>l.$$

Consequently, for $l \geq 2^k$

$$\sum_{l+1}^{2^{k+1}} \Pr\{I_n=1 \mid \mathcal{F}_{n-1}\} \leq K(2^{k+1}-l)\frac{1}{\sqrt{r(l)}} \leq K2^k\frac{1}{\sqrt{r(l)}}$$

and also the expected number of sites u_i which are successfully filled up in order during $[l+1, 2^{k+1}]$ is at most $K2^k(r(l))^{-1/2}$. A direct application of known exponential bounds, e.g. using Freedman (1973, theorem 4b) with

$$a=m=K2^{k+4}\frac{1}{\sqrt{r(l)}}+2^{k/2} \qquad b=K2^k\frac{1}{\sqrt{r(l)}} \leq \frac{m}{16}$$

now shows that conditionally on A_l

$$\Pr\{(12) \text{ occurs for the given path } u_t, \ldots, u_{t+m}\} \leq \Pr\{(17) \text{ occurs}\} \leq (\tfrac{1}{16}e)^m.$$

In view of (15), (16) and $r_l \leq l < 2^{k+1}$, it now follows that

$$\begin{aligned}\Pr\{r(2^{k+1})-r(l) \geq K2^{k+4}\frac{1}{\sqrt{r(l)}}+2^{k/2}+1\}\\ \leq \Pr\{(12) \text{ occurs for some path } u_t, \ldots, u_{t+m} \text{ satisfying (10) and (11)}\}\\ \leq 4\pi^2 2^{k+1}(\tfrac{4}{16}e)^m \leq 4\pi^2 2^{k+1}(\tfrac{1}{4}e)^{2^{k/2}}.\end{aligned} \tag{18}$$

The sum of the right-hand side of (18) over l in $2^k \leq l < 2^{k+1}$ and then over $k=1,2,\ldots$ is finite. This implies (9) by means of the Borel-Cantelli lemma (see Feller 1968, lemma VIII.3.1).

Step (iii). When $d \geq 3$ (8) has to be replaced by

$$\mu_{\partial A_n}(y) \leq Kn^{-1+2/d} \tag{19}$$

for each connected set A_n in z^d of n sites, and $y \in \partial A_n$. (19) is obtained from the identification of $\mu_B(y)$ in (5) as

$$e_B(y)\left(\sum_{z \in B} e_B(z)\right)^{-1} \tag{20}$$

where $e_B(y)$, the escape probability of B from y, is the probability that a random walk $S_0, S_1, \ldots$ starting from $S_0 = y$ never returns to B (see Spitzer 1976, prop. 26.2). The numerator in (20) is clearly ≤ 1, while Spitzer shows that the denominator equals the 'capacity' of B. Since the hitting distribution of ∂A_n is the same as that of $A_n \cup \partial A_n$, we find that

$$\mu_{\partial A_n}(y) \leq [\text{capacity of } (A_n \cup \partial A_n)]^{-1}$$

from which it is not hard to obtain (19). The proof of (2) from (19) is similar to step (ii).

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